

Explicit solutions for a $(2+1)$ -dimensional Toda-like chain.

V.E. Vekslerchik

Institute for Radiophysics and Electronics of NAS of Ukraine,
12, Proskura st., Kharkov, 61085, Ukraine

E-mail: vekslerchik@yahoo.com

Abstract. We consider a $(2+1)$ -dimensional Toda-like chain which can be viewed as a two-dimensional generalization of the Wu-Geng model and which is closely related to the two-dimensional Volterra, two-dimensional Toda and relativistic Toda lattices. In the framework of the Hirota direct approach, we present equations describing this model as a system of bilinear equations that belongs to the Ablowitz-Ladik hierarchy. Using the Jacobi-like determinantal identities and the Fay identity for the theta-functions, we derive its Toeplitz, dark-soliton and quasiperiodic solutions as well as the similar set of solutions for the two-dimensional Volterra chain.

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1. Introduction.

In this paper we consider a (2+1)-dimensional Toda-like chain

$$\Delta u_n = (\nabla u_n, \nabla u_n) \left(\frac{1}{u_n - u_{n+1}} + \frac{1}{u_n - u_{n-1}} \right), \quad (1.1)$$

where $u_n = u_n(x, y)$, Δ and ∇ are the two-dimensional Laplacian and gradient. This equation appears in the literature in various contexts. For example, it is known to describe the Bäcklund transformations of a Heisenberg-like magnetics (see [1]) and Laplace transformations of hydrodynamic-type systems in Riemann invariants [2]. In a recent paper [3] this system was shown to describe $O(3)$ σ -fields (three-component vectors σ_n of unit length) coupled by a nearest-neighbour Heisenberg-like interaction such as, *e.g.*, graphite-like magnetics when the spins inside one layer are governed by the Landau-Lifshitz theory with an effective Heisenberg interaction between adjacent layers: $\mathcal{H} = \mathcal{H}_{\text{LL}} + \mathcal{H}_{\text{H}}$ where $\mathcal{H}_{\text{LL}} = \sum_n \mathcal{E}_n$ with

$$\mathcal{E}_n = \int_{\mathbb{R}^2} dx dy (\nabla \sigma_n, \nabla \sigma_n) \quad (1.2)$$

and $\mathcal{H}_{\text{H}} = \frac{1}{2} \sum_n \sum_{p=n\pm 1} \mathcal{U}_{np}$ with

$$\mathcal{U}_{np} = g^2 \int_{\mathbb{R}^2} dx dy \ln \left(1 + (\sigma_n, \sigma_p) \right). \quad (1.3)$$

A remarkable feature of the model (1.1), which we will write in terms of the complex variables $z = x + iy$ and $\bar{z} = x - iy$,

$$\partial \bar{\partial} u_n = (\partial u_n) (\bar{\partial} u_n) \left(\frac{1}{u_n - u_{n+1}} + \frac{1}{u_n - u_{n-1}} \right) \quad (1.4)$$

where ∂ and $\bar{\partial}$ stand for $\partial/\partial z$ and $\partial/\partial \bar{z}$, is that it can be viewed as a connecting link between almost all Toda-like chains. In the following section, we present its relationships with the Wu-Geng chain (WGC) [4], the ‘two-dimensional Volterra equation’ (2DVE) [5, 6], the two-dimensional Toda lattice (2DTL) [7] and the relativistic Toda chain (RTC) [8, 9, 10, 11]

The main goal of this work is to obtain some explicit solutions for (1.4). We will not elaborate the inverse scattering transform or the algebro-geometric approach from scratch. Instead, we use the links between our equation and the 2DTL together with the results [12] that give us possibility of bilinearizing (1.4) and reducing it in section 3 to a system that belongs to the Ablowitz-Ladik hierarchy (ALH) [13]. Starting from the structure of the already known solutions for the ALH, we derive in sections 4 and 5 Toeplitz and dark-soliton solutions directly from some determinantal identities. In section 6, we use the Fay identity for the θ -functions to derive the quasiperiodic solutions. Finally, in section 7 the results of sections 4–6 are used to obtain the Toeplitz, dark-soliton and periodic solutions for the 2DVE.

2. WGC, 2DVE, 2DTL and RTC.

The main equations of this paper can be viewed as a straightforward generalization of the WGC to $(2+1)$ dimensions. Indeed, from equations (2) of [4],

$$u_{nt} = \frac{1}{v_{n+1}} - \frac{1}{v_n}, \quad v_{nt} = \frac{1}{u_n} - \frac{1}{u_{n-1}} \quad (2.1)$$

where u_{nt} stands for du_n/dt , one can obtain that functions w_n defined by $v_n = w_n - w_{n-1}$ satisfy

$$w_{nt} = \frac{1}{u_n} + C, \quad C = \text{constant} \quad (2.2)$$

which leads, after setting $C = 0$, to

$$w_{ntt} = w_{nt}^2 \frac{w_{n+1} - 2w_n + w_{n-1}}{(w_{n+1} - w_n)(w_n - w_{n-1})}. \quad (2.3)$$

Clearly, this equation coincides with (1.4) after replacing ∂ and $\bar{\partial}$ with d/dt .

On the other hand, rewriting equations (1.4) in terms of the variables

$$a_n = \frac{i\partial u_n}{u_{n+1} - u_n}, \quad b_n = \frac{-i\bar{\partial} u_n}{u_n - u_{n-1}} \quad (2.4)$$

one arrives at the system

$$\begin{cases} i\bar{\partial} a_n &= a_n (b_{n+1} - b_n) \\ i\partial b_n &= b_n (a_{n-1} - a_n) \end{cases} \quad (2.5)$$

which is closely related to the 2DTL: the quantities f_n defined by

$$f_n = \ln a_n b_n \quad (2.6)$$

satisfy

$$\partial\bar{\partial} f_n = e^{f_{n+1}} - 2e^{f_n} + e^{f_{n-1}}. \quad (2.7)$$

System (2.5) is known since the works of Leznov, Savel'ev and Smirnov [5, 6] where the authors demonstrated that this system, which was named the 'two-dimensional Volterra equation', represents the Bäcklund transformations for the 2DTL and constructed its general solutions in the finite case using the results of [14, 15].

Another model that we would like to discuss is the RTC [8, 9, 10, 11] which can be presented as a Hamiltonian system

$$i\partial q_n = \partial\mathcal{H}/\partial p_n, \quad i\partial p_n = -\partial\mathcal{H}/\partial q_n \quad (2.8)$$

with

$$\mathcal{H} = \sum_n e^{p_n} (e^{q_{n+1}-q_n} - 1). \quad (2.9)$$

Equations (2.8)

$$\begin{cases} i\partial q_n &= e^{p_n} (e^{q_{n+1}-q_n} - 1) \\ i\partial p_n &= e^{p_n+q_{n+1}-q_n} - e^{p_{n-1}+q_n-q_{n-1}} \end{cases} \quad (2.10)$$

rewritten in terms of the variables

$$a_n = e^{p_n + q_{n+1} - q_n}, \quad b_n = e^{-p_n} \quad (2.11)$$

are

$$\begin{cases} i\partial \ln a_n &= a_{n+1} - a_{n-1} - 1/b_{n+1} + 1/b_n \\ i\partial \ln b_n &= a_{n-1} - a_n. \end{cases} \quad (2.12)$$

One can easily note that the second of the above equations coincides with the second equation of (2.5). In a similar manner, starting from the Hamiltonian system

$$i\bar{\partial}q_n = \partial\bar{\mathcal{H}}/\partial p_n, \quad i\bar{\partial}p_n = -\partial\bar{\mathcal{H}}/\partial q_n \quad (2.13)$$

with

$$\bar{\mathcal{H}} = \sum_n e^{-p_n} (e^{q_n - q_{n-1}} - 1) \quad (2.14)$$

that has the form

$$\begin{cases} i\bar{\partial}q_n &= e^{-p_n} (1 - e^{q_n - q_{n-1}}) \\ i\bar{\partial}p_n &= e^{-p_{n+1} + q_{n+1} - q_n} - e^{-p_n + q_n - q_{n-1}} \end{cases} \quad (2.15)$$

one arrives, after using substituting (2.11), at the system

$$\begin{cases} i\bar{\partial} \ln a_n &= b_{n+1} - b_n \\ i\partial \ln b_n &= a_{n-1}b_{n-1}b_n - a_nb_nb_{n+1} \end{cases} \quad (2.16)$$

whose first equation is the first one from (2.5). The relationship between the two RTCs, (2.12) and (2.16), and our system (2.5) becomes transparent if one considers RTCs from the zero-curvature viewpoint based on the scattering problem [9, 10, 11]

$$a_n\psi_{n+1} - b_n^{-1}\psi_n = \Lambda(\psi_n - \psi_{n-1}). \quad (2.17)$$

System (2.12) is the compatibility condition of (2.17) and

$$i\partial\psi_n = a_n(\psi_{n+1} - \psi_n) \quad (2.18)$$

while (2.16) plays the same role for (2.17) and

$$i\bar{\partial}\psi_n = b_n(\psi_{n-1} - \psi_n). \quad (2.19)$$

As to our system, (2.5), it ensures the consistency of (2.18) taken together with (2.19),

$$\begin{cases} i\partial\psi_n &= a_n(\psi_{n+1} - \psi_n) \\ i\bar{\partial}\psi_n &= b_n(\psi_{n-1} - \psi_n). \end{cases} \quad (2.20)$$

To summarize, equations that are the subject of this paper can be viewed as describing the commutativity of two relativistic Toda flows. It should be noted that this commutativity leads, again, to the 2DTL. Indeed, considering equations (2.10) combined with (2.15) one can obtain by direct calculations that functions q_n satisfy

$$\partial\bar{\partial}q_n = \exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1}) \quad (2.21)$$

which is another form of the 2DTL.

Finally, comparing (2.4) with (2.20) one can conclude that solutions for our system, u_n , are nothing but solutions for the auxiliary linear problems for the 2DVE; in other words, the u_n -model is dual to (a_n, b_n) -model.

3. Bilinearization.

To bilinearize our equation, that can be written as

$$\begin{cases} \partial \bar{\partial} u_n &= p_n (u_{n+1} - 2u_n + u_{n-1}) \\ (\partial u_n) (\bar{\partial} u_n) &= p_n (u_{n+1} - u_n) (u_n - u_{n-1}) \end{cases} \quad (3.1)$$

we start with the 2DTL, assuming

$$p_n = \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2} \quad (3.2)$$

where τ_n is a solution for

$$\partial \bar{\partial} \ln \tau_n = \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2} \quad (3.3)$$

or, in the bilinear form,

$$D \bar{D} \tau_n \cdot \tau_n = 2 \tau_{n-1} \tau_{n+1} \quad (3.4)$$

where D and \bar{D} are the Hirota operators corresponding to ∂ and $\bar{\partial}$: $D a \cdot b = (\partial a) b - a (\partial b)$, $\bar{D} a \cdot b = (\bar{\partial} a) b - a (\bar{\partial} b)$.

One can easily obtain a wide range of solutions for the first equation of (3.1) by differentiating the 2DTL tau-function τ_n with respect to *any* parameter,

$$u_n = \frac{\partial}{\partial \zeta} \ln \tau_n. \quad (3.5)$$

Since τ_n can be considered as a solution for *all* equations of the 2DTL hierarchy, i.e. as a function of an infinite number of ‘times’, $\tau_n = \tau_n(t_1, t_2, \dots, \bar{t}_1, \bar{t}_2, \dots)$, $t_1 = z$, $\bar{t}_1 = \bar{z}$, the last formula can be generalized as follows:

$$u_n = \sum_{j=1}^{\infty} \left(c_j \frac{\partial \ln \tau_n}{\partial t_j} + \bar{c}_j \frac{\partial \ln \tau_n}{\partial \bar{t}_j} \right) \quad (3.6)$$

with arbitrary constants c_j and \bar{c}_j . However, this apparently most straightforward approach is rather hard to implement because the second equation of (3.1), which plays the role of the restriction for the set c_j and \bar{c}_j , is a bilinear system which we cannot solve. That is why we will use another way to deal with our equations. The key point, that will be proved below, is that u_n can be composed of *two* solutions for the 2DTL equation:

$$u_n = \frac{\omega_n}{\tau_n} \quad (3.7)$$

where

$$\frac{1}{2} D \bar{D} \omega_n \cdot \omega_n = \omega_{n-1} \omega_{n+1} + \lambda \omega_n^2 \quad (3.8a)$$

$$\frac{1}{2} D \bar{D} \tau_n \cdot \tau_n = \tau_{n-1} \tau_{n+1} + \lambda \tau_n^2 \quad (3.8b)$$

and λ is a unimportant constant that can be eliminated by multiplying ω_n and τ_n by $\exp(-\lambda z \bar{z})$ without modifying u_n . Calculating the first derivatives of u_n ,

$$\partial u_n = \frac{1}{\tau_n^2} D \omega_n \cdot \tau_n, \quad \bar{\partial} u_n = \frac{1}{\tau_n^2} \bar{D} \omega_n \cdot \tau_n \quad (3.9)$$

and its Laplacian,

$$\partial\bar{\partial}u_n = \frac{1}{\tau_n^2} D\bar{D} \omega_n \cdot \tau_n - \frac{u_n}{\tau_n^2} D\bar{D} \tau_n \cdot \tau_n \quad (3.10)$$

one can rewrite (3.1) as

$$\begin{cases} (D\bar{D} - 2\lambda) \omega_n \cdot \tau_n &= \tau_{n-1}\omega_{n+1} + \tau_{n+1}\omega_{n-1} \\ (D\omega_n \cdot \tau_n) (\bar{D}\omega_n \cdot \tau_n) &= (\tau_{n-1}\omega_n - \tau_n\omega_{n-1})(\tau_n\omega_{n+1} - \tau_{n+1}\omega_n). \end{cases} \quad (3.11)$$

The next step is to split the second equation of the above system in two bilinear equations. Introducing new τ -functions $\hat{\tau}_n$, $\hat{\omega}_n$, $\check{\tau}_n$ and $\check{\omega}_n$ by

$$iD\omega_n \cdot \tau_n = \hat{\tau}_n \hat{\omega}_n \quad (3.12a)$$

$$-i\bar{D}\omega_n \cdot \tau_n = \check{\tau}_n \check{\omega}_n \quad (3.12b)$$

and noting that the r.h.s. the second equation of (3.11) can be presented as $X_n X_{n-\delta}$ where

$$X_n = \tau_n \omega_{n+\delta} - \tau_{n+\delta} \omega_n, \quad \delta = \pm 1 \quad (3.13)$$

it is possible to achieve our goal by setting

$$\check{\omega}_n = \hat{\omega}_{n-\delta}, \quad \check{\tau}_n = \hat{\tau}_{n+\delta} \quad (3.14)$$

which leads to

$$\begin{cases} \tau_n \omega_{n+\delta} - \tau_{n+\delta} \omega_n &= \check{\tau}_n \hat{\omega}_n \\ iD\omega_n \cdot \tau_n &= \check{\tau}_{n-\delta} \hat{\omega}_n \\ -i\bar{D}\omega_n \cdot \tau_n &= \check{\tau}_n \hat{\omega}_{n-\delta}. \end{cases} \quad (3.15)$$

Now we have to close this system adding the equations describing the dependence of the new τ -functions on the variables z and \bar{z} . These equations should be (1) bilinear, (2) compatible and (3) imply as a consequence the first equation of (3.11). The resulting system can be written as the union of the ‘positive’ part,

$$\begin{cases} 0 &= iD\omega_n \cdot \tau_n - \check{\tau}_{n-\delta} \hat{\omega}_n \\ 0 &= iD\check{\tau}_n \cdot \tau_n + \check{\tau}_{n-\delta} \tau_{n+\delta} \\ 0 &= iD\omega_n \cdot \hat{\omega}_{n-\delta} + \omega_{n-\delta} \hat{\omega}_n, \end{cases} \quad (3.16)$$

the ‘negative’ one,

$$\begin{cases} 0 &= i\bar{D}\omega_n \cdot \tau_n + \check{\tau}_n \hat{\omega}_{n-\delta} \\ 0 &= i\bar{D}\check{\tau}_{n-\delta} \cdot \tau_n + \check{\tau}_n \tau_{n-\delta} \\ 0 &= i\bar{D}\omega_n \cdot \hat{\omega}_n + \omega_{n+\delta} \hat{\omega}_{n-\delta}, \end{cases} \quad (3.17)$$

combined with the restriction

$$0 = \tau_n \omega_{n+\delta} - \tau_{n+\delta} \omega_n - \check{\tau}_n \hat{\omega}_n. \quad (3.18)$$

One can verify that these bilinear equations indeed lead to the solution of our problem. As to their compatibility (that can be checked directly, but after rather long and tedious calculations), we would like to note that they are a part of the extended ALH [16].

Actually, we have passed from τ -functions of the 2DTL to ones of the ALH using the results of [12]. In other words, the above proceeding can be viewed as an explanation of the following fact, that can be verified by straightforward algebra: the compatible system (3.16)–(3.18) which belongs to the extended ALH provides solutions for the system (3.1). The reference to the ALH not only resolves the question of compatibility, but also gives us possibility of using the already known solutions for the Ablowitz-Ladik equations to build ones for the problem we are dealing with.

Definitions (2.4) and the above equations yield the following expressions for solutions of the 2DVE:

$$\delta = 1 : \quad a_n = \frac{\tilde{\tau}_{n-1}\tau_{n+1}}{\tilde{\tau}_n\tau_n}, \quad b_n = \frac{\tau_{n-1}\tilde{\tau}_n}{\tilde{\tau}_{n-1}\tau_n} \quad (3.19a)$$

$$\delta = -1 : \quad a_n = -\frac{\hat{\omega}_n\tau_{n+1}}{\tau_n\hat{\omega}_{n+1}}, \quad b_n = -\frac{\tau_{n-1}\hat{\omega}_{n+1}}{\tau_n\hat{\omega}_n} \quad (3.19b)$$

that will be used in section 7.

4. Toeplitz solutions.

This type of solutions can be constructed of the Toeplitz determinants

$$A_\ell^k = \det |\alpha_{k+a-b}|_{a,b=1,\dots,\ell}. \quad (4.1)$$

The main idea is that among various determinantal identities one can find ones that are similar to the equations we want to solve. In the appendix we derive the necessary formulae by applying the Jacobi identity to the determinants (4.1) and the framed ones,

$$F_{\ell+1}^k(\zeta) = \begin{vmatrix} 1 & \zeta & \zeta^2 & \dots & \zeta^\ell \\ \alpha_{k-1} & \alpha_k & \alpha_{k+1} & \dots & \alpha_{k+\ell-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix} \quad (4.2)$$

(see (A.28)–(A.33)). These formulae can then be rewritten in terms of the ℓ -order determinants B_ℓ^k , similar to (4.1), instead of the $(\ell+1)$ -order ones, $F_{\ell+1}^k(\zeta)$,

$$F_{\ell+1}^k(\zeta) = (-\zeta)^\ell B_\ell^{k-1}(\zeta), \quad (4.3)$$

where

$$B_\ell^k(\zeta) = \det |\beta_{k+a-b}(\zeta)|_{a,b=1,\dots,\ell} \quad (4.4)$$

with

$$\beta_k(\zeta) = \alpha_k - \zeta^{-1}\alpha_{k+1}. \quad (4.5)$$

The set of identities that we need to solve our equations is

$$0 = D_+ B_\ell^k \cdot A_\ell^k - \zeta^{-1} A_{\ell+1}^{k+1} B_{\ell-1}^{k-1} \quad (4.6a)$$

$$0 = D_+ B_\ell^k \cdot A_{\ell+1}^k + A_{\ell+1}^{k+1} B_\ell^{k-1} \quad (4.6b)$$

$$0 = D_+ B_\ell^k \cdot A_\ell^{k+1} - A_{\ell+1}^{k+1} B_{\ell-1}^k \quad (4.6c)$$

and

$$0 = D_- \mathbf{B}_\ell^k \cdot \mathbf{A}_\ell^k + \zeta^{-1} \mathbf{A}_{\ell+1}^k \mathbf{B}_{\ell-1}^k \quad (4.7a)$$

$$0 = D_- \mathbf{B}_\ell^k \cdot \mathbf{A}_\ell^{k+1} + \mathbf{A}_{\ell+1}^k \mathbf{B}_{\ell-1}^{k+1} \quad (4.7b)$$

$$0 = D_- \mathbf{B}_\ell^k \cdot \mathbf{A}_{\ell+1}^{k+1} + \mathbf{A}_{\ell+1}^k \mathbf{B}_\ell^{k+1} \quad (4.7c)$$

together with

$$0 = \mathbf{A}_\ell^{k+1} \mathbf{B}_\ell^k - \mathbf{A}_\ell^k \mathbf{B}_\ell^{k+1} + \zeta^{-1} \mathbf{A}_{\ell+1}^{k+1} \mathbf{B}_{\ell-1}^k. \quad (4.8)$$

Here, D_\pm stand for the Hirota operators corresponding to ∂_\pm and $\partial_\pm \alpha_k = \alpha_{k\pm 1}$.

Now, the problem of finding solutions for (3.16)–(3.18) becomes a ‘combinatorial’ one: one has to select τ_n , ω_n , $\check{\tau}_n$, $\hat{\omega}_n$, from \mathbf{A}_ℓ^k , \mathbf{B}_ℓ^k with different k and ℓ . Below, we obtain three families of solutions: infinite (with respect to n), semi-infinite and finite ones.

4.1. Infinite chain.

This type of solutions corresponds to the following choice of the τ -functions:

$$\tau_n = \tau_* \mathbf{A}_\ell^n, \quad \omega_n = \omega_* \mathbf{B}_\ell^n \quad (4.9)$$

and

$$\check{\tau}_n = \check{\tau}_* \mathbf{B}_{\ell-1}^n, \quad \hat{\omega}_n = \hat{\omega}_* \mathbf{A}_{\ell+1}^{n+1} \quad (4.10)$$

with constant τ_* , ω_* , $\check{\tau}_*$, $\hat{\omega}_*$. From (4.8), one can immediately derive that

$$0 = \tau_n \omega_{n+1} - \tau_{n+1} \omega_n - \check{\tau}_n \hat{\omega}_n \quad (4.11)$$

provided

$$\tau_* \omega_* = \zeta \check{\tau}_* \hat{\omega}_*. \quad (4.12)$$

Thus, our τ -functions solve (3.18) with $\delta = 1$. Furthermore, equations (4.6a)–(4.7c) lead to

$$0 = D_+ \omega_n \cdot \tau_n - \check{\tau}_{n-1} \hat{\omega}_n \quad (4.13a)$$

$$0 = D_+ \check{\tau}_n \cdot \tau_n + \check{\tau}_{n-1} \tau_{n+1} \quad (4.13b)$$

$$0 = D_+ \omega_n \cdot \hat{\omega}_{n-1} + \omega_{n-1} \hat{\omega}_n \quad (4.13c)$$

and

$$0 = D_- \omega_n \cdot \tau_n + \hat{\omega}_{n-1} \check{\tau}_n \quad (4.14a)$$

$$0 = D_- \check{\tau}_{n-1} \cdot \tau_n + \tau_{n-1} \check{\tau}_n \quad (4.14b)$$

$$0 = D_- \omega_n \cdot \hat{\omega}_n + \hat{\omega}_{n-1} \omega_{n+1}. \quad (4.14c)$$

It is clear that to complete solution for (3.16)–(3.17) one has to meet

$$\partial_+ = i\partial, \quad \partial_- = i\bar{\partial} \quad (4.15)$$

or, to take α_k to be solutions for the *linear* system

$$\begin{cases} i\partial\alpha_k &= \alpha_{k+1} \\ i\bar{\partial}\alpha_k &= \alpha_{k-1}. \end{cases} \quad (4.16)$$

The ‘symmetric’ set of solutions can be obtained by using, instead of (4.3), another representation of the determinants $F_{\ell+1}^k(\zeta)$:

$$F_{\ell+1}^k(\zeta) = C_\ell^k(\zeta) \quad (4.17)$$

where

$$C_\ell^k(\zeta) = \det |\gamma_{k+a-b}(\zeta)|_{a,b=1,\dots,\ell} \quad (4.18)$$

with

$$\gamma_k(\zeta) = \alpha_k - \zeta\alpha_{k-1}. \quad (4.19)$$

The calculations similar to the ones presented above demonstrate that τ -functions defined by

$$\tau_n = \tau_* A_\ell^n, \quad \omega_n = \omega_* C_\ell^n \quad (4.20)$$

and

$$\check{\tau}_n = \check{\tau}_* C_{\ell-1}^{n+1}, \quad \hat{\omega}_n = \hat{\omega}_* A_{\ell+1}^n \quad (4.21)$$

with

$$\zeta\tau_*\omega_* = -\check{\tau}_*\hat{\omega}_* \quad (4.22)$$

solve (3.16)–(3.18) with $\delta = 1$.

Both these sets of solutions can be written as

$$u_n(z, \bar{z}) = u_* \frac{\det |\alpha_{n+a-b}^\pm(z, \bar{z})|_{a,b=1,\dots,\ell}}{\det |\alpha_{n+a-b}(z, \bar{z})|_{a,b=1,\dots,\ell}} \quad (4.23)$$

where ℓ is an arbitrary positive integer and the elements of the determinants are given by

$$\alpha_k(z, \bar{z}) = \int_\Gamma dh \hat{\alpha}(h) h^k \exp[-i\Theta_h(z, \bar{z})] \quad (4.24a)$$

$$\alpha_k^\pm(z, \bar{z}) = \int_\Gamma dh \hat{\alpha}(h) [1 - (h/\zeta)^{\pm 1}] h^k \exp[-i\Theta_h(z, \bar{z})] \quad (4.24b)$$

(we replaced β_k and γ_k with α_k^\pm) with arbitrary contour Γ , function $\hat{\alpha}(h)$ and constant u_* . The ‘dispersion law’ $\Theta_h(z, \bar{z})$ is given by

$$\Theta_h(z, \bar{z}) = hz + h^{-1}\bar{z}. \quad (4.25)$$

As an example, let us consider one of the simplest solutions of (4.16). Noting that system (4.16) leads to the Helmholtz equation, $\partial\bar{\partial}\alpha_k + \alpha_k = 0$, and rewriting the latter using the polar coordinates, $z = re^{i\theta}$, one can obtain in a standard way

$$\alpha_k = \exp[-i(\theta + \pi/2)k] J_k(2r) \quad (4.26)$$

where J_k is the k th Bessel function. These solutions correspond to (4.24a) and (4.24b) with $\hat{\alpha}(h) = (2\pi i h)^{-1}$ and Γ being the unit circumference: $\Gamma = \{h : |h| = 1\}$. In the ‘elementary’ case of $\ell = 1$ expression (4.23) can be written as

$$u_n = u_* + u_{\pm} \frac{J_{n\pm 1}(2r)}{J_n(2r)} e^{\mp i\theta} \quad (4.27)$$

where u_* and u_{\pm} ($u_{\pm} = \pm i u_* \zeta^{\mp 1}$) are arbitrary constants. It is easy to see that solutions we have obtained are complex and singular, which, however, does not mean that they are non-physical. Say in the case of [3], that was discussed in the introduction, the spin components, $\sigma_n = (\sigma_n^{(1)}, \sigma_n^{(2)}, \sigma_n^{(3)})$, are related to u_n by $\sigma_n^{(1)} + i\sigma_n^{(2)} = 2u_n/(1 + |u_n|^2)$ and $\sigma_n^{(3)} = (1 - |u_n|^2)/(1 + |u_n|^2)$. Thus, in the general case ($\sigma_n^{(2)} \neq 0$) u_n is complex and singularities of u_n correspond to the vertical (southward) orientation of σ_n : $u_n = \infty \Leftrightarrow \sigma_n^{(1)} = \sigma_n^{(2)} = 0$ and $\sigma_n^{(3)} = -1$.

4.2. Semi-infinite chain.

This type of solutions appears if one identify the index n with the size of the determinants A_{ℓ}^k, B_{ℓ}^k . Consider the functions $T_n, W_n, \check{T}_n, \hat{W}_n$ given by

$$T_n = T_* A_n^k, \quad W_n = W_* B_n^k \quad (4.28)$$

and

$$\check{T}_n = \check{T}_* B_n^{k-1}, \quad \hat{W}_n = \hat{W}_* A_{n+1}^{k+1} \quad (4.29)$$

with constants $T_*, W_*, \check{T}_*, \hat{W}_*$ being related by

$$T_* W_* = -\zeta \check{T}_* \hat{W}_*. \quad (4.30)$$

It is straightforward to verify that they solve

$$0 = T_n W_{n+1} - T_{n+1} W_n - \check{T}_n \hat{W}_n \quad (4.31)$$

as well as

$$\begin{aligned} 0 &= D_+ W_n \cdot T_n + \check{T}_{n-1} \hat{W}_n \\ 0 &= D_+ \check{T}_n \cdot T_n - \check{T}_{n-1} T_{n+1} \\ 0 &= D_+ W_n \cdot \hat{W}_{n-1} - W_{n-1} \hat{W}_n \end{aligned} \quad (4.32)$$

and

$$\begin{aligned} 0 &= (D_- + \zeta^{-1}) W_n \cdot T_n + \hat{W}_{n-1} \check{T}_n \\ 0 &= (D_- + \zeta^{-1}) \check{T}_{n-1} \cdot T_n + T_{n-1} \check{T}_n \\ 0 &= (D_- + \zeta^{-1}) W_n \cdot \hat{W}_n + \hat{W}_{n-1} W_{n+1} \end{aligned} \quad (4.33)$$

that almost coincide with (3.16)–(3.18) for $\delta = 1$ after identifying

$$\partial = i\partial_+, \quad \bar{\partial} = -i\partial_-. \quad (4.34)$$

The extra terms can be eliminated by $\exp(i\bar{z}/\zeta)$ factor and one arrives at the following solutions:

$$\tau_n = \tau_* \exp(i\bar{z}/\zeta) A_n^k, \quad \omega_n = \omega_* B_n^k \quad (4.35)$$

and

$$\tilde{\tau}_n = \tilde{\tau}_* \mathbf{B}_n^{k-1}, \quad \hat{\omega}_n = \hat{\omega}_* \exp(i\bar{z}/\zeta) \mathbf{A}_{n+1}^{k+1} \quad (4.36)$$

where τ_* , ω_* , $\tilde{\tau}_*$, $\hat{\omega}_*$ are arbitrary constants and

$$\zeta = -\frac{\tau_* \omega_*}{\tilde{\tau}_* \hat{\omega}_*}. \quad (4.37)$$

In a similar way one can derive the ‘symmetric’ set of solutions using the determinants \mathbf{C}_n^k . These two sets of the Toeplitz solutions u_n can be written as

$$u_n(z, \bar{z}) = u_* e^{i\phi_{\pm}(z, \bar{z})} \frac{\det |\alpha_{k+a-b}^{\pm}(z, \bar{z})|_{a,b=1,\dots,n}}{\det |\alpha_{k+a-b}(z, \bar{z})|_{a,b=1,\dots,n}} \quad (4.38)$$

with an arbitrary positive integer k . The elements of the determinants are given again by (4.24a) and (4.24b) with the ‘dispersion law’

$$\Theta_h(z, \bar{z}) = -hz + h^{-1}\bar{z} \quad (4.39)$$

while the phases ϕ_{\pm} are given by

$$\phi_+(z, \bar{z}) = -\bar{z}/\zeta, \quad \phi_-(z, \bar{z}) = \zeta z. \quad (4.40)$$

The above formulae for the Toeplitz solutions demonstrate the typical for integrable systems phenomenon: solutions for the *nonlinear* equations are determinants of matrices that satisfy *linear* ones. This rule in our case can be extended as follows: the relationships between the τ -functions τ_n and ω_n (which are, recall, solutions for the 2DTL used to construct u_n) become *linear* when rewritten in terms of the ‘inside-determinant’ objects, α_k and α_k^{\pm} .

As in the case of the infinite chain, let us consider one of the simplest solutions for (4.34). Using the polar coordinates $z = re^{i\theta}$ one can obtain

$$\alpha_k = \exp[-i(\theta + \pi/2)k] I_k(2r) \quad (4.41)$$

where I_k are the modified Bessel functions. This leads, together with definition (4.40) of ϕ_{\pm} ,

$$\phi_{\pm}(r, \theta) = \mp r \zeta^{\mp 1} e^{\mp i\theta}, \quad (4.42)$$

to

$$\alpha_k^{\pm} = \exp[-i(\theta + \pi/2)k] \left[I_k(2r) + \frac{\phi_{\pm}(r, \theta)}{ir} I_{k\pm 1}(2r) \right] \quad (4.43)$$

and consequently to

$$u_n = u_* e^{i\phi_{\pm}(r, \theta)} \frac{\det \left| I_{k+a-b}(2r) + \frac{\phi_{\pm}(r, \theta)}{ir} I_{k+a-b\pm 1}(2r) \right|_{a,b=1,\dots,n}}{\det |I_{k+a-b}(2r)|_{a,b=1,\dots,n}} \quad (4.44)$$

In context of the Heisenberg-like model [3] these solutions describe circular magnetic domain structures.

4.3. Finite chain.

The semi-infinite solutions derived in the previous subsection can be easily modified to provide the finite ones with $u_n \neq 0$ for $n = 0, 1, \dots, N$ only. To this end one has to define

$$u_0(z, \bar{z}) = u_* e^{i\phi_{\pm}(z, \bar{z})}, \quad (4.45)$$

where ϕ_{\pm} are defined by (4.40), and to replace the integrals in (4.24a) and (4.24b) with finite sums,

$$\alpha_k(z, \bar{z}) = \sum_{p=1}^N \hat{\alpha}_p h_p^k \exp[-i\Theta_{h_p}(z, \bar{z})] \quad (4.46a)$$

$$\alpha_k^{\pm}(z, \bar{z}) = \sum_{p=1}^N \hat{\alpha}_p [1 - (h_p/\zeta)^{\pm 1}] h_p^k \exp[-i\Theta_{h_p}(z, \bar{z})] \quad (4.46b)$$

where

$$\Theta_{h_p}(z, \bar{z}) = -h_p z + h_p^{-1} \bar{z}. \quad (4.47)$$

These modifications do not change the fact that $i\partial\alpha_k = -\alpha_{k+1}$ and $i\bar{\partial}\alpha_k = \alpha_{k-1}$ which implies that u_0 and u_n given by (4.38) still solve (1.4) for $n = 1, \dots, N-1$. Thus, it remains to prove that they solve (1.4) for $n = 0, N$ as well. The case $n = 0$ is trivial because for both choices of ϕ_{\pm}

$$\partial\bar{\partial}u_0 = (\partial u_0)(\bar{\partial}u_0) = 0 \quad (4.48)$$

converting equation (1.4) with $n = 0$ into trivial one. Considering the right-end equation, one can show, using the identity

$$\det \left| \sum_{p=1}^N \alpha_p x_p^{a-b} \right|_{a,b=1,\dots,N} = (-1)^{\frac{1}{2}N(N-1)} \prod_{p=1}^N \frac{\alpha_p}{x_p^{N-1}} \prod_{1 \leq p < q \leq N} (x_p - x_q)^2, \quad (4.49)$$

that

$$u_N(z, \bar{z}) = u'_* e^{i\phi_{\pm}(z, \bar{z})}, \quad (4.50)$$

where u'_* is a constant: the factors $\exp[-i\Theta_{h_p}(z, \bar{z})]$ cancel themselves when we divide $\det |\alpha_{k+a-b}^{\pm}|$ by $\det |\alpha_{k+a-b}|$ leading to

$$u'_* = u_* \prod_{p=1}^N [1 - (h_p/\zeta)^{\pm 1}] \quad (4.51)$$

Hence,

$$\partial\bar{\partial}u_N = (\partial u_N)(\bar{\partial}u_N) = 0, \quad (4.52)$$

ensuring solution of (1.4) for $n = N$.

To summarize, $N+1$ functions, u_0 given by (4.45) and

$$u_n(z, \bar{z}) = u_0(z, \bar{z}) \frac{\det |\alpha_{k+a-b}^{\pm}(z, \bar{z})|_{a,b=1,\dots,n}}{\det |\alpha_{k+a-b}(z, \bar{z})|_{a,b=1,\dots,n}}, \quad n = 1, \dots, N \quad (4.53)$$

with (4.46a)–(4.47) solve the system of $N+1$ equations (1.4) for $n = 0, \dots, N$.

5. Soliton solutions.

The dark-soliton solutions for our equations can be constructed of the determinants

$$\Omega(A) = \det |1 + A| \quad (5.1)$$

of the $N \times N$ matrices A that satisfy the ‘almost rank-1’ condition

$$LA - AR = |\ell\rangle\langle a|. \quad (5.2)$$

Here 1 is the $N \times N$ unit matrix, L and R are the constant diagonal matrices, $|\ell\rangle$ is the constant N -component column, $|\ell\rangle = (\ell_1, \dots, \ell_N)^T$ and $\langle a|$ is the N -component row depending on the coordinates, $\langle a(z, \bar{z})| = (a_1(z, \bar{z}), \dots, a_N(z, \bar{z}))$. The second part of the ‘solitonic ansatz’, except for (5.2), is that the dependence of A on all coordinates $(z, \bar{z}$ and n) can be built by means of the shifts

$$\mathbb{T}_\zeta \Omega = \Omega (A H_\zeta) \quad (5.3)$$

with the matrices H_ζ being defined by

$$H_\zeta = (L - \zeta)(R - \zeta)^{-1} \quad (5.4)$$

(we do not write the unit matrix explicitly, so $(L - \zeta)$ stands for $(L - \zeta 1)$ etc).

The remarkable property of the above matrices is that the determinants

$$\Omega_\zeta = \mathbb{T}_\zeta \Omega, \quad \Omega_{\xi\eta} = \mathbb{T}_\xi \mathbb{T}_\eta \Omega \quad (5.5)$$

satisfy the Fay-like identity

$$(\xi - \eta) \Omega_\zeta \Omega_{\xi\eta} + (\eta - \zeta) \Omega_\xi \Omega_{\eta\zeta} + (\zeta - \xi) \Omega_\eta \Omega_{\zeta\xi} = 0 \quad (5.6)$$

(see, *e.g.*, appendix of [17]). Introducing the differential operators ∂_ξ defined by

$$\mathbb{T}_\xi^{-1} \mathbb{T}_{\xi+\delta} \Omega = \Omega + i\delta \partial_\xi \Omega + O(\delta^2) \quad (5.7)$$

one can derive from (5.6) the differential Fay identities

$$(\xi - \alpha) i D_\xi \Omega_\alpha \cdot \Omega = (\mathbb{T}_\xi^{-1} \Omega_\alpha) (\mathbb{T}_\xi \Omega) - \Omega_\alpha \Omega \quad (5.8)$$

and

$$(\xi - \alpha)(\xi - \beta) i D_\xi \Omega_\alpha \cdot \Omega_\beta = (\alpha - \beta) [(\mathbb{T}_\xi^{-1} \Omega_{\alpha\beta}) (\mathbb{T}_\xi \Omega) - \Omega_\alpha \Omega_\beta]. \quad (5.9)$$

Applying these identities two times, one arrives at

$$\frac{1}{2} (\xi - \eta)^2 D_\xi D_\eta \Omega \cdot \Omega = \Omega^2 - (\mathbb{T}_\xi \mathbb{T}_\eta^{-1} \Omega) (\mathbb{T}_\xi^{-1} \mathbb{T}_\eta \Omega) \quad (5.10)$$

from which it is easy to obtain solutions for the 2DTL by associating the $\partial, \bar{\partial}$ operators with ∂_ξ and ∂_η (with fixed ξ and η) and introducing the n -variable by means of the powers of \mathbb{T}_ν

$$\Omega_n = \mathbb{T}_\nu^n \Omega \quad (5.11)$$

with

$$H_\xi = H_\eta H_\nu. \quad (5.12)$$

From the Fay identities (5.6) and (5.9) one can derive after straightforward calculations that the functions

$$T_n = T_* h_\alpha^n \mathbb{T}_\nu^n \Omega_\alpha \quad (5.13a)$$

$$W_n = W_* h_\beta^n \mathbb{T}_\nu^n \Omega_\beta \quad (5.13b)$$

$$\tilde{T}_n = \tilde{T}_* \check{h}^n \mathbb{T}_\nu^n \mathbb{T}_\eta^{-1} \Omega_{\alpha\beta} \quad (5.13c)$$

$$\hat{W}_n = \hat{W}_* \hat{h}^n \mathbb{T}_\nu^n \Omega_\xi \quad (5.13d)$$

where the h -factors are defined by

$$h_\alpha = \frac{\xi - \alpha}{\eta - \alpha}, \quad h_\beta = \frac{\xi - \beta}{\eta - \beta}, \quad (5.14)$$

$$\check{h} = \mu h_\alpha h_\beta (\xi - \eta), \quad \hat{h} = \bar{\mu} (\xi - \eta) \quad (5.15)$$

(with constants μ and $\bar{\mu}$ satisfying $\mu\bar{\mu} = (\xi - \eta)^{-2}$) and the constants T_* , W_* , \tilde{T}_* , \hat{W}_* are related by

$$\tilde{T}_* \hat{W}_* = T_* W_* \frac{(\beta - \alpha)(\xi - \eta)}{(\alpha - \eta)(\beta - \eta)} \quad (5.16)$$

satisfy the following equations:

$$[iD_\xi + \lambda_\xi(\beta, \alpha)] W_n \cdot T_n = \mu \tilde{T}_{n-1} \hat{W}_n \quad (5.17a)$$

$$[iD_\xi + \lambda_\xi(\beta, \eta)] \tilde{T}_n \cdot T_n = -\mu \tilde{T}_{n-1} T_{n+1} \quad (5.17b)$$

$$[iD_\xi + \lambda_\xi(\beta, \eta)] W_n \cdot \hat{W}_{n-1} = -\mu W_{n-1} \hat{W}_n \quad (5.17c)$$

and

$$[iD_\eta + \lambda_\eta(\beta, \alpha)] W_n \cdot T_n = \bar{\mu} \hat{W}_{n-1} \tilde{T}_n \quad (5.18a)$$

$$[iD_\eta + \lambda_\eta(\beta, \xi)] \tilde{T}_{n-1} \cdot T_n = \bar{\mu} T_{n-1} \tilde{T}_n \quad (5.18b)$$

$$[iD_\eta + \lambda_\eta(\beta, \xi)] W_n \cdot \hat{W}_n = \bar{\mu} \hat{W}_{n-1} W_{n+1}. \quad (5.18c)$$

Here

$$\lambda_\xi(\alpha, \beta) = \lambda_\xi(\alpha) - \lambda_\xi(\beta) \quad (5.19)$$

with

$$\lambda_\xi(\gamma) = \frac{1}{\xi - \gamma}. \quad (5.20)$$

Comparing the above equations with (3.16) and (3.17) it is easy to conclude that to obtain solutions for our problem one has to identify

$$\partial = \mu^{-1} \partial_\xi \quad \bar{\partial} = -\bar{\mu}^{-1} \partial_\eta \quad (5.21)$$

and to introduce some linear in z and \bar{z} phases to eliminate extra λ -terms in (5.17a)–(5.18c). This leads to the following expression for u_n :

$$u_n(z, \bar{z}) = u_* e^{i\phi(z, \bar{z})} h^n \frac{\det |1 + A_n(z, \bar{z}) H_\beta|}{\det |1 + A_n(z, \bar{z}) H_\alpha|} \quad (5.22)$$

where

$$h = \frac{h_\beta}{h_\alpha}, \quad (5.23)$$

$$A_n(z, \bar{z}) = A(z, \bar{z}) H_\nu^n \quad (5.24)$$

and

$$\phi = \mu^{-1} \lambda_\xi(\alpha, \beta) z + \bar{\mu}^{-1} \lambda_\eta(\beta, \alpha) \bar{z} + \text{constant}. \quad (5.25)$$

Finally, it remains to resolve the restriction (5.12) and to write down explicitly the dependence of A on the coordinates, which can be done by calculating the limit in (5.7),

$$iA^{-1} \partial_\zeta A = (L - R) (L - \zeta)^{-1} (R - \zeta)^{-1} \quad (5.26)$$

(for any ζ).

The restriction (5.12) implies that the matrices L and R are not independent,

$$(L - \xi) (R - \xi) = (\nu - \xi)(\eta - \xi) 1. \quad (5.27)$$

Introducing matrices \hat{L} and \hat{R} ,

$$\hat{L} = \frac{1}{\xi - \nu} (L - \nu), \quad \hat{R} = \frac{1}{\xi - \nu} (R - \nu) \quad (5.28)$$

satisfying

$$(\hat{L} - 1) (\hat{R} - 1) = \frac{\xi - \eta}{\xi - \nu} 1 \quad (5.29)$$

and parameter f , instead of μ and $\bar{\mu}$,

$$f = \frac{1}{\mu(\xi - \eta)} = \bar{\mu}(\xi - \eta) \quad (5.30)$$

the z - and \bar{z} -dependence of A and ϕ can be presented in a symmetric form as

$$A(z, \bar{z}) = A_* \exp \{iM(z, \bar{z})\} \quad (5.31)$$

where A_* is a constant matrix,

$$M(z, \bar{z}) = f (\hat{R} - \hat{L}) z + f^{-1} (\hat{R}^{-1} - \hat{L}^{-1}) \bar{z}, \quad (5.32)$$

and

$$\phi(z, \bar{z}) = f (h_\beta^{-1} - h_\alpha^{-1}) z + f^{-1} (h_\beta - h_\alpha) \bar{z}. \quad (5.33)$$

These formulae together with (5.22) describe the N -dark-soliton solutions for our problem.

The simplest (one-soliton) solution can be written as

$$u_n(z, \bar{z}) = u_* e^{2i\varphi_n(z, \bar{z})} [1 + \tanh \rho \tanh U_n(z, \bar{z})] \quad (5.34)$$

where

$$\rho = \frac{1}{2} \ln \frac{1 - h_\alpha \hat{L}}{1 - h_\alpha \hat{R}} \frac{1 - h_\beta \hat{R}}{1 - h_\beta \hat{L}}, \quad (5.35)$$

(note that now \hat{L} and \hat{R} are just complex numbers) the phase φ_n is given by

$$\varphi_n(z, \bar{z}) = \xi_0 z + \eta_0 \bar{z} + \zeta_0 n \quad (5.36)$$

with

$$\xi_0 = \frac{f}{2} (h_\beta^{-1} - h_\alpha^{-1}), \quad \eta_0 = \frac{1}{2f} (h_\beta - h_\alpha) \quad (5.37)$$

and

$$\sin^2 \zeta_0 = \xi_0 \eta_0, \quad (5.38)$$

while U_n is given by

$$U_n(z, \bar{z}) = \xi_s z + \eta_s \bar{z} + \zeta_s n \quad (5.39)$$

with

$$\xi_s = \frac{if}{2} (\hat{R} - \hat{L}), \quad \eta_s = \frac{i}{2f} (\hat{R}^{-1} - \hat{L}^{-1}) \quad (5.40)$$

and

$$\sinh^2 \zeta_s = \xi_s \eta_s. \quad (5.41)$$

To conclude, we would like to note that in the soliton case the link between τ_n and ω_n is *linear* in the terms of the matrices A_n : the matrices that appear in the determinants in (5.22) are related by the constant diagonal matrix $H_\beta H_\alpha^{-1}$.

6. Quasiperiodic solutions.

In this section, we derive the periodic solutions for our equation proceeding in the way similar to the one used in the previous section. The main difference is in the starting point: instead of identity (5.6) we use the original Fay's trisecant identity (see (6.7) below).

The solutions that we derive below are combinations of the θ -functions defined over a compact Riemann surface Γ of the genus g for which one can choose in a standard way a set of closed contours (cycles) $\{a_i, b_i\}_{i=1, \dots, g}$ with the intersection indices

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij} \quad i, j = 1, \dots, g \quad (6.1)$$

and g independent holomorphic differentials ϖ_k satisfying the normalization conditions

$$\oint_{a_i} \varpi_k = \delta_{ik}, \quad i, k = 1, \dots, g. \quad (6.2)$$

The matrix of the b -periods,

$$\Omega_{ik} = \oint_{b_i} \varpi_k, \quad i, k = 1, \dots, g \quad (6.3)$$

determines the so-called period lattice, $L_\Omega = \{\mathbf{m} + \Omega \mathbf{n}, \quad \mathbf{m}, \mathbf{n} \in \mathbb{Z}^g\}$, and the Abel mapping from Γ to the $2g$ torus \mathbb{C}^g / L_Ω (the Jacobian of this surface),

$$P \rightarrow \int_{P_0}^P \boldsymbol{\omega} \quad (6.4)$$

where P is a point of Γ , $\boldsymbol{\omega}$ is the g -vector of the 1-forms, $\boldsymbol{\omega} = (\varpi_1, \dots, \varpi_g)^T$, and P_0 is some fixed point of Γ .

The θ -function, $\theta(\boldsymbol{\zeta}) = \theta(\boldsymbol{\zeta}, \Omega)$, is defined by

$$\theta(\boldsymbol{\zeta}) = \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp \left\{ \pi i \left(\mathbf{n}, \Omega \mathbf{n} \right) + 2\pi i \left(\mathbf{n}, \boldsymbol{\zeta} \right) \right\} \quad (6.5)$$

where $(\mathbf{n}, \boldsymbol{\zeta})$ stands for $\sum_{i=1}^g n_i \zeta_i$. This is a quasiperiodic function on \mathbb{C}^g

$$\theta(\boldsymbol{\zeta} + \mathbf{n}) = \theta(\boldsymbol{\zeta}) \quad (6.6a)$$

$$\theta(\boldsymbol{\zeta} + \Omega \mathbf{n}) = \exp \left\{ -\pi i \left(\mathbf{n}, \Omega \mathbf{n} \right) - 2\pi i \left(\mathbf{n}, \boldsymbol{\zeta} \right) \right\} \theta(\boldsymbol{\zeta}) \quad (6.6b)$$

for any $\mathbf{n} \in \mathbb{Z}^g$.

The calculations presented below are based on the famous Fay's trisecant formula [18, 19] that can be written as

$$\varepsilon_B^P \varepsilon_A^Q \theta_A^P \theta_B^Q - \varepsilon_A^P \varepsilon_B^Q \theta_B^P \theta_A^Q + \varepsilon_Q^P \varepsilon_B^A \theta \theta_{AB}^{PQ} = 0. \quad (6.7)$$

Here

$$\theta_{P_1 \dots P_m}^{Q_1 \dots Q_m} = \theta \left(\boldsymbol{\zeta} + \sum_{i=1}^m \int_{P_i}^{Q_i} \boldsymbol{\omega} \right) \quad (6.8)$$

and the skew-symmetric function ε_P^Q , $\varepsilon_P^Q = -\varepsilon_Q^P$, is given by

$$\varepsilon_P^Q = \theta \left(\mathbf{e} + \int_P^Q \boldsymbol{\omega} \right) \quad (6.9)$$

where \mathbf{e} is a zero of the θ -function: $\theta(\mathbf{e}) = 0$.

Now let us define the differential operators ∂_X by

$$\theta_X^P = \theta + \varepsilon_X^P \partial_X \theta + o(\varepsilon_X^P). \quad (6.10)$$

In what follows, we use ∂_A and ∂_B defined near two points A and B that are fixed. One of them will play the role of ∂ and another of $\bar{\partial}$. Taking the limit in (6.7) one can obtain the differential Fay's identity

$$[D_X + \lambda_X(P, Q)] \theta_Q^P \cdot \theta = \gamma_X(P, Q) \theta_X^P \theta_Q^X \quad (6.11)$$

where D_X is the Hirota operator corresponding to ∂_X and the functions $\lambda_X(P)$ and $\gamma_X(P, Q)$ are defined by

$$\lambda_X(P, Q) = \lambda_X(P) - \lambda_X(Q) \quad (6.12)$$

with

$$\lambda_X(P) = \lim_{Y \rightarrow X} \frac{\varepsilon_Y^P - \varepsilon_X^P}{\varepsilon_X^Y \varepsilon_X^P} \quad (6.13)$$

and

$$\gamma_X(P, Q) = \frac{\varepsilon_Q^P}{\varepsilon_X^P \varepsilon_X^Q}. \quad (6.14)$$

Consider the functions

$$T_n = T_* [h(P)]^n \theta \left(\zeta_n + \int_A^P \omega \right) \quad (6.15a)$$

$$W_n = W_* [h(Q)]^n \theta \left(\zeta_n + \int_A^Q \omega \right) \quad (6.15b)$$

$$\check{T}_n = \check{T}_* \check{h}^n \theta \left(\zeta_n + \int_{AB}^{PQ} \omega \right) \quad (6.15c)$$

$$\hat{W}_n = \hat{W}_* \hat{h}^n \theta(\zeta_n) \quad (6.15d)$$

where

$$h(P) = \frac{\varepsilon_A^P}{\varepsilon_B^P} \quad (6.16)$$

and the n -dependence is given by

$$\zeta_{n+1} = \zeta_n + \int_B^A \omega. \quad (6.17)$$

It follows from (6.7) that if one imposes the restrictions

$$\check{T}_* \hat{W}_* = T_* W_* \frac{\varepsilon_Q^P \varepsilon_A^B}{\varepsilon_B^P \varepsilon_B^Q} \quad (6.18)$$

and

$$\check{h} \hat{h} = h(P) h(Q), \quad (6.19)$$

then these functions satisfy

$$T_n W_{n+1} - T_{n+1} W_n = \check{T}_n \hat{W}_n. \quad (6.20)$$

Furthermore, by taking

$$\check{h} = \mu_A \varepsilon_A^B \frac{\varepsilon_A^P \varepsilon_A^Q}{\varepsilon_B^P \varepsilon_B^Q}, \quad \hat{h} = \frac{1}{\mu_A \varepsilon_A^B} \quad (6.21)$$

where μ_A and μ_B are two constants related by

$$\mu_A \mu_B (\varepsilon_A^B)^2 = 1 \quad (6.22)$$

one can get from (6.11) the set of identities which will be associated with the ∂ -flows:

$$[D_A + \lambda_A(Q, P)] W_n \cdot T_n = -\mu_A \check{T}_{n-1} \hat{W}_n \quad (6.23a)$$

$$[D_A + \lambda_A(Q, B)] \check{T}_n \cdot T_n = \mu_A \check{T}_{n-1} T_{n+1} \quad (6.23b)$$

$$[D_A + \lambda_A(Q, B)] W_n \cdot \hat{W}_{n-1} = \mu_A W_{n-1} \hat{W}_n \quad (6.23c)$$

and another one,

$$[D_B + \lambda_B(Q, P)] W_n \cdot T_n = -\mu_B \hat{W}_{n-1} \check{T}_n \quad (6.24a)$$

$$[D_B + \lambda_B(Q, A)] \check{T}_{n-1} \cdot T_n = -\mu_B T_{n-1} \check{T}_n \quad (6.24b)$$

$$[D_B + \lambda_B(Q, A)] W_n \cdot \hat{W}_n = -\mu_B \hat{W}_{n-1} W_{n+1} \quad (6.24c)$$

associated with the $\bar{\partial}$ -equations.

Comparing the above equations with (3.16) and (3.17), one arrives at

$$\partial = \frac{i}{\mu_A} \partial_A, \quad \bar{\partial} = \frac{1}{i\mu_B} \partial_B \quad (6.25)$$

which determines the dependence on z and \bar{z} ,

$$\zeta_n(z, \bar{z}) = \zeta_* + z\mathbf{a} + \bar{z}\mathbf{b} + n\mathbf{c} \quad (6.26)$$

where \mathbf{c} was defined above,

$$\mathbf{c} = \int_B^A \boldsymbol{\omega}, \quad (6.27)$$

while

$$\mathbf{a} = if \lim_{P \rightarrow A} \frac{\varepsilon_B^P}{\varepsilon_A^P} \int_A^P \boldsymbol{\omega} \quad (6.28a)$$

$$\mathbf{b} = if^{-1} \lim_{P \rightarrow B} \frac{\varepsilon_A^P}{\varepsilon_B^P} \int_B^P \boldsymbol{\omega} \quad (6.28b)$$

where we have introduced the constant f instead of μ_A and μ_B ,

$$f = \frac{1}{\mu_A \varepsilon_B^A} = \mu_B \varepsilon_B^A. \quad (6.29)$$

Again, the λ -terms in (6.23a)–(6.24c) can be eliminated by adding linear in z and \bar{z} phases, which leads to

$$u_n = \frac{W_n}{T_n} e^{i\phi} \quad (6.30)$$

where

$$\phi(z, \bar{z}) = f \varepsilon_A^B \lambda_A(P, Q) z + f^{-1} \varepsilon_B^A \lambda_B(P, Q) \bar{z} + \text{constant}. \quad (6.31)$$

Using the definitions of W_n and T_n , and introducing $h = h(Q)/h(P)$,

$$h = \frac{\varepsilon_A^Q \varepsilon_B^P}{\varepsilon_A^P \varepsilon_B^Q}, \quad (6.32)$$

we can write the final expression for the quasiperiodic solutions as

$$u_n(z, \bar{z}) = u_* e^{i\phi(z, \bar{z})} h^n \frac{\theta\left(\zeta_n(z, \bar{z}) + \int_A^Q \boldsymbol{\omega}\right)}{\theta\left(\zeta_n(z, \bar{z}) + \int_A^P \boldsymbol{\omega}\right)}. \quad (6.33)$$

This time, the link between ω_n and τ_n is *linear* in terms of ζ_n : the transformation $\tau_n \rightarrow \omega_n$ is achieved by $\zeta_n \rightarrow \zeta_n + \int_P^Q \boldsymbol{\omega}$.

The simplest of the quasiperiodic solutions (6.33), after some slight modifications, can be rewritten as a cnoidal wave:

$$u_n(z, \bar{z}) = u_* e^{i\phi_n(z, \bar{z})} \text{sn } \zeta_n(z, \bar{z}) \quad (6.34)$$

where the phase ϕ_n and the function ζ_n are given by

$$\phi_n(z, \bar{z}) = \xi_0 z + \eta_0 \bar{z} + \delta_0 n, \quad (6.35a)$$

$$\zeta_n(z, \bar{z}) = \xi_p z + \eta_p \bar{z} + \delta_p n \quad (6.35b)$$

and $\operatorname{sn} z = \operatorname{sn}(z, k)$ is the elliptic sine. The parameters $\xi_{0,p}$, $\eta_{0,p}$, $\delta_{0,p}$ and the elliptic modulus k are related by

$$\xi_p \eta_p = \operatorname{sn}^2 \delta_p \quad (6.36a)$$

$$\xi_0 \eta_p + \xi_p \eta_0 = 2 \sin \delta_0 \operatorname{sn} \delta_p \quad (6.36b)$$

and

$$\xi_0 \eta_0 = |\operatorname{dn} \delta_p - e^{i\delta_0} \operatorname{cn} \delta_p|^2 \quad (6.36c)$$

for real δ_0 and δ_p .

7. Solutions of the two-dimensional Volterra equation.

As was mentioned in section 2, the authors of [5, 6] derived general solutions of the 2DVE in the case of finite chain. Here we would like to present several other classes of solutions for this system, namely ones that can be obtained from the results presented in this paper using (3.19a) and (3.19b).

7.1. Toeplitz solutions.

As follows from (3.19a) and (3.19b), the constants τ_* and $\tilde{\tau}_*$ as well as the phases \bar{z}/ζ (that we defined in (4.9), (4.10) and (4.35), (4.36)) disappear from the final formulae for a_n and b_n , that can be written as

$$a_n = \frac{A_\ell^{n+1} B_{\ell-1}^{n-1}}{A_\ell^n B_{\ell-1}^n}, \quad b_n = \frac{A_\ell^{n-1} B_{\ell-1}^n}{A_\ell^n B_{\ell-1}^{n-1}} \quad (7.1)$$

in the infinite case ($-\infty < n < \infty$) and

$$a_n = \frac{A_{n+1}^k B_{n-1}^{k-1}}{A_n^k B_n^{k-1}}, \quad b_n = \frac{A_{n-1}^k B_n^{k-1}}{A_n^k B_{n-1}^{k-1}} \quad (7.2)$$

in the semi-infinite case ($1 \leq n < \infty$). One can easily obtain similar solutions with C_ℓ^n - and C_n^k -determinants which we do not write here.

7.2. Soliton solutions.

Making the shift $A_n \rightarrow A_n H_\xi H_\alpha^{-1}$ and introducing the matrix B_n ,

$$B_n = A_n H_\beta \quad (7.3)$$

one can presents soliton solutions for (2.5) in the following form:

$$a_n = c_* \frac{\det |1 + A_n H_\xi H_\nu| \det |1 + B_n|}{\det |1 + A_n H_\xi| \det |1 + B_n H_\nu|} \quad (7.4a)$$

$$b_n = \frac{1}{c_*} \frac{\det |1 + A_n H_\eta| \det |1 + B_n H_\nu|}{\det |1 + A_n H_\eta H_\nu| \det |1 + B_n|} \quad (7.4b)$$

with $c_* = f/h_\beta$ and matrices $A_n = A_n(z, \bar{z})$ and H_ζ defined in section 5. In these expressions one can find a symmetry $1/b_n = a_n(\xi \rightarrow \eta)$ which is a manifestation of many symmetries inherent in the 2DVE.

7.3. Periodic solutions.

It follows from (3.19a) and (3.19b) that the phases that one has to introduce to eliminate the λ -terms in (6.23a)–(6.24c) cancel themselves and formulae (6.15a), (6.15c) yield *periodic* solutions for the 2DVE. After the shift $\zeta_n \rightarrow \zeta_n - \int_A^P \omega$ one can write these solutions as

$$a_n = c_* \frac{\theta\left(\zeta_n + \int_B^A \omega\right) \theta\left(\zeta_n + \int_A^Q \omega\right)}{\theta(\zeta_n) \theta\left(\zeta_n + \int_B^Q \omega\right)} \quad (7.5a)$$

$$b_n = \frac{1}{c_*} \frac{\theta\left(\zeta_n + \int_A^B \omega\right) \theta\left(\zeta_n + \int_B^Q \omega\right)}{\theta(\zeta_n) \theta\left(\zeta_n + \int_A^Q \omega\right)} \quad (7.5b)$$

where $c_* = f\varepsilon_Q^B/\varepsilon_A^Q$ with $\zeta_n(z, \bar{z})$ and f defined in section 6. One can see that of two points that parametrize solutions (6.33), P and Q , only one is left and, again, one can find the symmetry linking a_n and b_n : $b_n \propto a_n (A \leftrightarrow B)$.

8. Conclusion

We have obtained three types of explicit solutions for the model (1.1). To conclude, we would like to discuss a few questions that have not been studied comprehensively in this paper. The first one is related to whether these solutions are real or complex. All presented above are, in general, complex ones. From the viewpoint of applications there are situations when namely such solutions are needed. For example, in the model of the graphite-like magnetics with Heisenberg-type interaction discussed in [3] $u_n \propto \sigma_n^{(1)} + i\sigma_n^{(2)}$ where $\sigma_n^{(k)}$ are the components of σ_n , $\sigma_n = (\sigma_n^{(1)}, \sigma_n^{(2)}, \sigma_n^{(3)})$, which means that in general situation u_n is complex. However, sometimes one may be interested in the real solutions, as *e.g.*, in the theory of hydrodynamic-type systems [2]. So, one faces a natural question of restrictions that should be imposed on the parameters of solutions that ensure their reality. This is a particular case of a problem which frequently arises in the theory of integrable equations, especially in the case of (quasi)periodic solutions, and surely deserves special studies.

Another range of questions is related to the symmetries of our problem. The symmetry $\delta \rightarrow -\delta$ in (3.13) and the following equations, which is another form of $n \rightarrow -n$ symmetry, is the simplest one. The symmetries that we saw in the previous section, $1/b_n = a_n(\xi \rightarrow \eta)$ and $b_n \propto a_n (A \leftrightarrow B)$ are less trivial. At the level of the ALH, they describe the links between various Miwa's shifts. Since the ALH is out of the scope of this paper (though implicitly we used some of its properties), we do not discuss this topic further, noting only that studies in this direction may lead to some kind of nonlinear superposition formulae for our problem.

Finally, we would like to repeat that from the viewpoint of the 2DVE or RTC our main result, solutions of (1.1), are nothing but solutions of the auxiliary linear problems

(2.17)–(2.19). In other words, after some additional work (such as, *e.g.*, extracting the dependence on Λ) one can obtain from our results the Baker-Akhiezer function of the 2DVE or RTC, which is usually more complicated problem than to obtain solutions for an equation itself and which can be useful for applications.

Appendix A. Toeplitz determinants.

Applying the Jacobi identity

$$\Delta \Delta_{k_1 k_2}^{j_1 j_2} = \Delta_{k_1}^{j_1} \Delta_{k_2}^{j_2} - \Delta_{k_2}^{j_1} \Delta_{k_1}^{j_2} \quad (\text{A.1})$$

where Δ is the determinant of a matrix, Δ_k^j is the determinants of the same matrix with j th row and k th column being excluded, *etc.*, to $\mathbf{F}_{\ell+1}^k(\zeta)$ for $(j_1, j_2; k_1, k_2)$ being equal to $(1, \ell+1; 1, \ell+1)$, $(1, 2; 1, \ell+1)$, $(1, 3; 1, \ell+1)$ and $(1, \ell; 1, \ell+1)$ one can obtain

$$\mathbf{X}_\ell^k := \mathbf{A}_{\ell-1}^k \mathbf{F}_{\ell+1}^k(\zeta) - \mathbf{A}_\ell^k \mathbf{F}_\ell^k(\zeta) + \zeta \mathbf{A}_\ell^{k-1} \mathbf{F}_\ell^{k+1}(\zeta) = 0 \quad (\text{A.2})$$

$$\mathbf{Z}_\ell^k := \mathbf{A}_{\ell-1}^k \mathbf{F}_{\ell+1}^{k+1}(\zeta) - \mathbf{A}_\ell^{k+1} \mathbf{F}_\ell^k(\zeta) + \zeta \mathbf{A}_\ell^k \mathbf{F}_\ell^{k+1}(\zeta) = 0 \quad (\text{A.3})$$

$$\mathbf{I}_\ell^k := \mathbf{F}_{\ell+1}^{k+1}(\zeta) \partial_+ \mathbf{A}_{\ell-1}^k - \mathbf{A}_\ell^{k+1} \partial_+ \mathbf{F}_\ell^k(\zeta) + \zeta \mathbf{A}_\ell^k \partial_+ \mathbf{F}_\ell^{k+1}(\zeta) = 0 \quad (\text{A.4})$$

$$\bar{\mathbf{I}}_\ell^k := \mathbf{F}_{\ell+1}^k(\zeta) \partial_- \mathbf{A}_{\ell-1}^k - \mathbf{A}_\ell^k \partial_- \mathbf{F}_\ell^k(\zeta) + \zeta \mathbf{A}_\ell^{k-1} \partial_- \mathbf{F}_\ell^{k+1}(\zeta) = 0 \quad (\text{A.5})$$

correspondingly. These identities, combined with

$$\Delta_\ell^k := (\mathbf{A}_\ell^k)^2 - \mathbf{A}_{\ell-1}^k \mathbf{A}_{\ell+1}^k - \mathbf{A}_\ell^{k-1} \mathbf{A}_\ell^{k+1} = 0 \quad (\text{A.6})$$

lead to

$$\mathbf{W}_\ell^k := \mathbf{A}_\ell^k \mathbf{F}_{\ell+1}^k - \mathbf{A}_{\ell+1}^k \mathbf{F}_\ell^k - \mathbf{A}_\ell^{k-1} \mathbf{F}_{\ell+1}^{k+1} = 0 \quad (\text{A.7})$$

$$\mathbf{Y}_\ell^k := \mathbf{A}_\ell^k \mathbf{F}_{\ell+1}^{k+1} - \mathbf{A}_\ell^{k+1} \mathbf{F}_{\ell+1}^k + \zeta \mathbf{A}_{\ell+1}^k \mathbf{F}_\ell^{k+1} = 0. \quad (\text{A.8})$$

Expanding the above equations,

$$\mathbf{W}_\ell^k = \Delta_\ell^k + \zeta \bar{\mathbf{w}}_\ell^k + \dots + (-\zeta)^{\ell-1} \mathbf{w}_\ell^k \quad (\text{A.9})$$

$$\mathbf{X}_\ell^k = \zeta \bar{\mathbf{x}}_\ell^k + \dots + (-\zeta)^{\ell-1} \mathbf{x}_\ell^k \quad (\text{A.10})$$

$$\mathbf{Y}_\ell^k = \zeta \bar{\mathbf{y}}_\ell^k + \dots + (-\zeta)^\ell \Delta_\ell^k \quad (\text{A.11})$$

$$\mathbf{Z}_\ell^k = \zeta \bar{\mathbf{z}}_\ell^k + \dots + (-\zeta)^{\ell-1} \mathbf{z}_\ell^k, \quad (\text{A.12})$$

where

$$\mathbf{w}_\ell^k = D_+ \mathbf{A}_\ell^{k-1} \cdot \mathbf{A}_\ell^k - \mathbf{A}_{\ell-1}^{k-1} \mathbf{A}_{\ell+1}^k \quad (\text{A.13})$$

$$\mathbf{x}_\ell^k = D_+ \mathbf{A}_\ell^{k-1} \cdot \mathbf{A}_{\ell-1}^k - \mathbf{A}_{\ell-1}^{k-1} \mathbf{A}_\ell^k \quad (\text{A.14})$$

$$\mathbf{z}_\ell^k = D_+ \mathbf{A}_\ell^k \cdot \mathbf{A}_{\ell-1}^k - \mathbf{A}_{\ell-1}^{k-1} \mathbf{A}_\ell^{k+1} \quad (\text{A.15})$$

with

$$\bar{\mathbf{w}}_\ell^k = -\mathbf{A}_\ell^k \partial_- \mathbf{A}_\ell^k + \mathbf{A}_{\ell+1}^k \partial_- \mathbf{A}_{\ell-1}^k + \mathbf{A}_\ell^{k-1} \partial_- \mathbf{A}_\ell^{k+1} \quad (\text{A.16})$$

$$\bar{\mathbf{x}}_\ell^k = D_- \mathbf{A}_{\ell-1}^k \cdot \mathbf{A}_\ell^k + \mathbf{A}_\ell^{k-1} \mathbf{A}_{\ell-1}^{k+1} \quad (\text{A.17})$$

$$\bar{\mathbf{y}}_\ell^k = D_- \mathbf{A}_\ell^k \cdot \mathbf{A}_\ell^{k+1} + \mathbf{A}_{\ell-1}^{k+1} \mathbf{A}_{\ell+1}^k \quad (\text{A.18})$$

$$\bar{\mathbf{z}}_\ell^k = D_- \mathbf{A}_{\ell-1}^k \cdot \mathbf{A}_\ell^{k+1} + \mathbf{A}_{\ell-1}^{k+1} \mathbf{A}_\ell^k, \quad (\text{A.19})$$

and introducing

$$\mathbf{J}_\ell^k := \partial_+ \mathbf{Z}_\ell^k - \mathbf{I}_\ell^k = 0 \quad (\text{A.20})$$

$$\bar{\mathbf{J}}_\ell^k := \partial_- \mathbf{X}_\ell^k - \bar{\mathbf{I}}_\ell^k = 0 \quad (\text{A.21})$$

one can derive by straightforward algebra the following identities:

$$\mathbf{A}_{\ell-1}^k (D_+ \mathbf{F}_{\ell+1}^{k+1} \cdot \mathbf{A}_\ell^k + \mathbf{A}_{\ell+1}^{k+1} \mathbf{F}_\ell^k) = \mathbf{A}_\ell^k \mathbf{J}_\ell^k - (\partial_+ \mathbf{A}_\ell^k) \mathbf{Z}_\ell^k - \mathbf{F}_\ell^k \mathbf{w}_\ell^{k+1} \quad (\text{A.22})$$

$$\mathbf{A}_{\ell-1}^{k-1} (D_+ \mathbf{F}_{\ell+1}^k \cdot \mathbf{A}_\ell^k + \zeta \mathbf{A}_{\ell+1}^k \mathbf{F}_\ell^k) = \mathbf{A}_\ell^k \mathbf{J}_\ell^{k-1} - (\partial_+ \mathbf{A}_\ell^k) \mathbf{Z}_\ell^{k-1} - \zeta \mathbf{F}_\ell^k \mathbf{w}_\ell^k \quad (\text{A.23})$$

$$\begin{aligned} \mathbf{A}_{\ell-2}^k (D_+ \mathbf{F}_\ell^{k+1} \cdot \mathbf{A}_\ell^k + \mathbf{A}_\ell^{k+1} \mathbf{F}_\ell^k) &= \mathbf{A}_\ell^k \mathbf{J}_{\ell-1}^k - (\partial_+ \mathbf{A}_\ell^k) \mathbf{Z}_{\ell-1}^k + \mathbf{A}_\ell^{k+1} \mathbf{X}_{\ell-1}^k \\ &\quad - \mathbf{F}_{\ell-1}^k \mathbf{x}_\ell^{k+1} + \zeta \mathbf{F}_{\ell-1}^{k+1} \mathbf{Z}_\ell^k \end{aligned} \quad (\text{A.24})$$

and

$$\mathbf{A}_{\ell-1}^{k+1} (D_- \mathbf{F}_{\ell+1}^{k+1} \cdot \mathbf{A}_\ell^k - \mathbf{A}_{\ell+1}^k \mathbf{F}_\ell^{k+1}) = \mathbf{A}_\ell^k \bar{\mathbf{J}}_\ell^{k+1} - (\partial_- \mathbf{A}_\ell^k) \mathbf{X}_\ell^{k+1} - \mathbf{F}_\ell^{k+1} \bar{\mathbf{y}}_\ell^k \quad (\text{A.25})$$

$$\mathbf{A}_{\ell-1}^k (D_- \mathbf{F}_{\ell+1}^k \cdot \mathbf{A}_\ell^k - \zeta \mathbf{A}_{\ell+1}^{k-1} \mathbf{F}_\ell^{k+1}) = \mathbf{A}_\ell^k \bar{\mathbf{J}}_\ell^k - (\partial_- \mathbf{A}_\ell^k) \mathbf{X}_\ell^k - \zeta \mathbf{F}_\ell^{k+1} \bar{\mathbf{y}}_\ell^{k-1} \quad (\text{A.26})$$

$$\begin{aligned} \mathbf{A}_{\ell-2}^k (D_- \mathbf{F}_\ell^k \cdot \mathbf{A}_\ell^k + \mathbf{A}_\ell^{k-1} \mathbf{F}_\ell^{k+1}) &= \mathbf{A}_\ell^k \bar{\mathbf{J}}_{\ell-1}^k + \mathbf{A}_\ell^{k-1} \mathbf{Z}_{\ell-1}^k - (\partial_- \mathbf{A}_\ell^k) \mathbf{X}_{\ell-1}^k \\ &\quad + \mathbf{F}_{\ell-1}^k \bar{\mathbf{x}}_\ell^k - \zeta \mathbf{F}_{\ell-1}^{k+1} \bar{\mathbf{Z}}_\ell^{k-1} \end{aligned} \quad (\text{A.27})$$

which means

$$D_+ \mathbf{F}_{\ell+1}^{k+1} \cdot \mathbf{A}_\ell^k + \mathbf{A}_{\ell+1}^{k+1} \mathbf{F}_\ell^k = 0 \quad (\text{A.28})$$

$$D_+ \mathbf{F}_{\ell+1}^k \cdot \mathbf{A}_\ell^k + \zeta \mathbf{A}_{\ell+1}^k \mathbf{F}_\ell^k = 0 \quad (\text{A.29})$$

$$D_+ \mathbf{F}_\ell^{k+1} \cdot \mathbf{A}_\ell^k + \mathbf{A}_\ell^{k+1} \mathbf{F}_\ell^k = 0 \quad (\text{A.30})$$

and

$$D_- \mathbf{F}_{\ell+1}^{k+1} \cdot \mathbf{A}_\ell^k - \mathbf{A}_{\ell+1}^k \mathbf{F}_\ell^{k+1} = 0 \quad (\text{A.31})$$

$$D_- \mathbf{F}_{\ell+1}^k \cdot \mathbf{A}_\ell^k - \zeta \mathbf{A}_{\ell+1}^{k-1} \mathbf{F}_\ell^{k+1} = 0 \quad (\text{A.32})$$

$$D_- \mathbf{F}_\ell^k \cdot \mathbf{A}_\ell^k + \mathbf{A}_\ell^{k-1} \mathbf{F}_\ell^{k+1} = 0. \quad (\text{A.33})$$

These equations, when rewritten in terms of \mathbf{B}_ℓ^k given by (4.3), are nothing but (4.6a)–(4.7c), while in terms of \mathbf{C}_ℓ^k given by (4.17) they become

$$0 = D_+ \mathbf{C}_\ell^k \cdot \mathbf{A}_\ell^k + \zeta \mathbf{C}_{\ell-1}^k \mathbf{A}_{\ell+1}^k \quad (\text{A.34})$$

$$0 = D_+ \mathbf{C}_\ell^{k+1} \cdot \mathbf{A}_\ell^k + \mathbf{A}_{\ell+1}^{k+1} \mathbf{C}_{\ell-1}^k \quad (\text{A.35})$$

$$0 = D_+ \mathbf{C}_\ell^{k+1} \cdot \mathbf{A}_{\ell+1}^k + \mathbf{A}_{\ell+1}^{k+1} \mathbf{C}_\ell^k \quad (\text{A.36})$$

and

$$0 = D_- \mathbf{C}_\ell^k \cdot \mathbf{A}_\ell^k - \zeta \mathbf{A}_{\ell+1}^{k-1} \mathbf{C}_{\ell-1}^{k+1} \quad (\text{A.37})$$

$$0 = D_- \mathbf{C}_\ell^k \cdot \mathbf{A}_{\ell+1}^k + \mathbf{A}_{\ell+1}^{k-1} \mathbf{C}_\ell^{k+1} \quad (\text{A.38})$$

$$0 = D_- \mathbf{C}_\ell^{k+1} \cdot \mathbf{A}_\ell^k - \mathbf{A}_{\ell+1}^k \mathbf{C}_{\ell-1}^{k+1}. \quad (\text{A.39})$$

The above formulae are enough to obtain the Toeplitz solutions presented in section 4.

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